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An Approximation Method to Solve Einstein's Field Equations on a Curved Background Metric

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Abstract

An approximation method is developed to calculate the gravitational field of a matter source $T_{\mu\nu}$ moving on a curved background metric that is an exact solution of the field equations and deviates only weakly from flat space-time. The field $h_{\mu\nu}$ of the source $T_{\mu\nu}$ is supposed to be much smaller than the curved part of the background, so that in the series expansion of $h_{\mu\nu}$ each order can be expanded in powers of the background.

1. *Introduction*

The approximation methods hitherto developed to solve Einstein's field equations are based on a series expansion of small deviations from flat space-time; see perhaps Einstein et al. (1928), Bertotti and Plebanski (1960), Das et al. (1961), Havas and Goldberg (1962) and other references given by Havas and Goldberg. That means that in the lowest order of approximation the metric has the form $g_{\mu\nu} = \eta_{\mu\nu}$. In many cases, however, it is suitable to consider the motion of a matter source $T_{\mu\nu}$ on a curved background metric that is unchanged by the motion of the source $T_{\mu\nu}$. According to this it shall be supposed that in the total metric field

$$g_{\mu\nu} = \overset{(\text{in})}{g}_{\mu\nu} + h_{\mu\nu} \tag{1.1}$$

the part $h_{\mu\nu}$ of the source $T_{\mu\nu}$ is much smaller than the curved part of the background metric $\overset{(\text{in})}{g}_{\mu\nu}$, i.e.,

$$|\overset{(\text{in})}{g}_{\mu\nu} - \eta_{\mu\nu}| \gg |h_{\mu\nu}| \tag{1.2a}$$

Furthermore it is assumed that $\overset{(\mathrm{in})}{g}_{\mu\nu}$ is an *exact* solution of the field equations and deviates only weakly from flat space-time, i.e.,

$$|\overset{(\mathrm{in})}{g}_{\mu\nu} - \eta_{\mu\nu}| \ll 1 \tag{1.2b}$$

Finally the assumption is made that the orders of magnitude expressed in equations (1.2) are conserved at partial derivatives. On the basis of these assumptions in the present paper an approximation method is developed for expanding the field $h_{\mu\nu}$ into series.

For the case of a spherically symmetric background Peters (1966) had developed a method to find an approximate solution. However, this method linearizes in the perturbing field $h_{\mu\nu}$ and beyond it considers in the background only terms linear in the gravitational potential Φ of the central mass. The method proposed in this paper takes into consideration the full curved part $\overset{(\mathrm{in})}{\gamma}_{\mu\nu} = \overset{(\mathrm{in})}{g}_{\mu\nu} - \eta_{\mu\nu}$ of any background with $|\overset{(\mathrm{in})}{\gamma}_{\mu\nu}| \ll 1$ and allows one to calculate all orders of the perturbing field $h_{\mu\nu}$.

2. *Field Equations*

Using the metric in the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \overset{(\mathrm{in})}{\gamma}_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu} \tag{2.1a}$$

$$g^{\mu\nu} = \eta^{\mu\nu} + \overset{(\mathrm{in})}{\sigma}{}^{\mu\nu} + k^{\mu\nu} = \eta^{\mu\nu} + \sigma^{\mu\nu} \tag{2.1b}$$

with

$$(\eta_{\mu\lambda} + \overset{(\mathrm{in})}{\gamma}_{\mu\lambda})(\eta^{\lambda\nu} + \overset{(\mathrm{in})}{\sigma}{}^{\lambda\nu}) = \overset{(\mathrm{in})}{g}_{\mu\lambda}\overset{(\mathrm{in})}{g}{}^{\lambda\nu} = \delta^{\nu}{}_{\mu} \tag{2.1c}$$

Einstein's field equations for the total field take the form

$$\begin{aligned}\Box\gamma_{\mu\nu} &+ \gamma_{|\mu|\nu} - \gamma^{\alpha}{}_{\mu|\alpha|\nu} - \gamma^{\alpha}{}_{\nu|\alpha|\mu} - \eta_{\mu\nu}(\Box\gamma - \gamma^{\alpha\beta}{}_{|\beta|\alpha}) \\ &= -2\kappa_0(\overset{(\mathrm{in})}{T}_{\mu\nu} + T_{\mu\nu}) - 2\Theta_{\mu\nu}\end{aligned} \tag{2.2a}$$

(raising and lowering of indices always by $\eta^{\mu\nu}$ or $\eta_{\mu\nu}$, vertical lines signify partial derivative, $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$). Here $\overset{(\mathrm{in})}{T}_{\mu\nu}$ is the source of the background field and $\gamma = \gamma_{\mu\nu}\eta^{\mu\nu}$. $\Theta_{\mu\nu}$ represents the nonlinear part of the field equations, i.e.,

$$\begin{aligned}\Theta_{\mu\nu} = {}& \sigma^{\alpha\beta}(\partial_\nu\Gamma_{\alpha,\beta\mu} - \partial_\beta\Gamma_{\alpha,\mu\nu}) + g^{\alpha\beta}g^{\lambda\kappa}(\Gamma_{\alpha,\lambda\kappa}\Gamma_{\beta,\mu\nu} \\ &- \Gamma_{\alpha,\lambda\mu}\Gamma_{\beta,\kappa\nu}) - \tfrac{1}{2}g_{\mu\nu}g^{\alpha\beta}\sigma^{\rho\tau}(\partial_\tau\Gamma_{\alpha,\beta\rho} - \partial_\beta\Gamma_{\alpha,\rho\tau}) \\ &- \tfrac{1}{2}g_{\mu\nu}g^{\alpha\beta}g^{\lambda\kappa}g^{\rho\tau}(\Gamma_{\alpha,\lambda\kappa}\Gamma_{\beta,\rho\tau} - \Gamma_{\alpha,\lambda\rho}\Gamma_{\beta,\kappa\tau}) \\ &- \tfrac{1}{2}\eta^{\rho\sigma}(g_{\mu\nu}\sigma^{\alpha\beta} + \gamma_{\mu\nu}\eta^{\alpha\beta})(\partial_\sigma\Gamma_{\alpha,\beta\rho} - \partial_\beta\Gamma_{\alpha,\rho\sigma})\end{aligned} \tag{2.2b}$$

The form (2.1a), (2.1b) of the metric and the resulting field equations (2.2) are invariant only under coordinate transformations that transform $\eta_{\mu\nu}$ into $\eta_{\mu\nu}$. That means that the metric in the representation (2.1a), (2.1b) and the field equations in the form of equations (2.2) restrict the four-dimensional coordinate transformations to Lorentz transformations. The background field had been supposed to be an exact solution of the field equations, i.e.,

$$\Box \overset{(\text{in})}{\gamma}_{\mu\nu} + \overset{(\text{in})}{\gamma}_{|\mu|\nu} - \overset{(\text{in})}{\gamma}{}^{\alpha}{}_{\mu|\alpha|\nu} - \overset{(\text{in})}{\gamma}{}^{\alpha}{}_{\nu|\alpha|\mu} - \eta_{\mu\nu}(\Box \overset{(\text{in})}{\gamma} - \overset{(\text{in})}{\gamma}{}^{\alpha\beta}{}_{|\beta|\alpha})$$
$$= -2\kappa_0 \overset{(\text{in})}{T}_{\mu\nu} - 2\Theta_{\mu\nu}(\overset{(\text{in})}{g}_{\lambda\kappa}) \tag{2.3}$$

Then it follows from equations (2.2) and (2.3) that

$$\Box h_{\mu\nu} + h_{|\mu|\nu} - h^{\alpha}{}_{\mu|\alpha|\nu} - h^{\alpha}{}_{\nu|\alpha|\mu} - \eta_{\mu\nu}(\Box h - h^{\alpha\beta}{}_{|\beta|\alpha})$$
$$= -2\kappa_0 T_{\mu\nu} - 2\vartheta_{\mu\nu} \tag{2.4a}$$

with

$$\vartheta_{\mu\nu} = \Theta_{\mu\nu} - \Theta_{\mu\nu}(\overset{(\text{in})}{g}_{\lambda\kappa}) \tag{2.4b}$$

To simplify the field equations (2.4) one can choose a suitable gauge. However, such a gauge has to be chosen in a form that is invariant under Lorentz transformations. In the following the gauge

$$\partial_\nu \psi_\mu{}^\nu = 0, \qquad \psi_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}h \tag{2.5}$$

is assumed. This gauge is invariant under Lorentz transformations. It is also suitable since the left-hand side of the field equations (2.4a) can be written in the form

$$\Box \psi_{\mu\nu} - \partial_\nu \partial_\alpha \psi_\mu{}^a - \partial_\mu \partial_\alpha \psi_\nu{}^\alpha + \eta_{\mu\nu}\partial_\alpha \partial_\beta \psi^{\alpha\beta}$$

so that the gauge (2.5) leads to

$$\Box \psi_{\mu\nu} = -2\kappa_0 T_{\mu\nu} - 2\vartheta_{\mu\nu} \tag{2.6}$$

The field equations (2.6) in connection with the gauge (2.5) are equivalent to the field equations (2.4).

3. *Conservation Laws*

The divergence of the left-hand side $L_{\mu\nu}$ of the field equations (2.2a) vanishes, i.e., $\partial_\nu L_\mu{}^\nu = 0$. From this it follows that

$$\partial_\nu \{\kappa_0 \overset{(\text{tot})}{T}_\mu{}^\nu + \Theta_\mu{}^\nu\} = 0 \tag{3.1a}$$

with

$$\overset{(\text{tot})}{T}_{\mu\nu} = \overset{(\text{in})}{T}_{\mu\nu} + T_{\mu\nu} \tag{3.1b}$$

The conservation law (3.1) is equivalent to

$$\partial_\nu \{\kappa_0 \overset{(\mathrm{tot})}{\mathscr{T}}{}_\mu{}^\nu - t_\mu{}^\nu\} = 0 \tag{3.2a}$$

with

$$\overset{(\mathrm{tot})}{\mathscr{T}}{}_\mu{}^\nu = g^{\nu\lambda}(-g)^{1/2}\overset{(\mathrm{tot})}{T}_{\mu\lambda}, \qquad g = \det g_{\mu\nu} \tag{3.2b}$$

and Einstein's energy-momentum complex

$$t_\mu{}^\nu = \tfrac{1}{2}(-g)^{1/2}\{[\Gamma^\nu_{\rho\sigma}(g^{\lambda\rho}g^{\tau\sigma} - \tfrac{1}{2}g^{\lambda\tau}g^{\rho\sigma}) - \Gamma^\rho_{\rho\sigma}(g^{\lambda\nu}g^{\tau\sigma} - \tfrac{1}{2}g^{\lambda\tau}g^{\nu\sigma})]g_{\lambda\tau|\mu} - \delta_\mu{}^\nu g^{\sigma\tau}(\Gamma^\rho_{\lambda\sigma}\Gamma^\lambda_{\rho\tau} - \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\sigma\tau})\} \tag{3.2c}$$

Analogously the conservation law

$$\partial_\nu \{\kappa_0 \overset{(\mathrm{in})}{T}_\mu{}^\nu + \overset{(\mathrm{in})}{\Theta}_\mu{}^\nu(g_{\lambda\kappa})\} = 0 \tag{3.3}$$

which follows from equation (2.3) is equivalent to

$$\partial_\nu \{\kappa_0 \overset{(\mathrm{in})}{\mathscr{T}}{}_\mu{}^\nu - \overset{(\mathrm{in})}{t}_\mu{}^\nu(g_{\lambda\kappa})\} = 0 \tag{3.4a}$$

with

$$\overset{(\mathrm{in})}{\mathscr{T}}{}_\mu{}^\nu = \overset{(\mathrm{in})}{g}{}^{\nu\lambda}(-\overset{(\mathrm{in})}{g})^{1/2}\overset{(\mathrm{in})}{T}_{\mu\lambda} \tag{3.4b}$$

From equations (3.1)–(3.4) one can conclude that the conservation law

$$\partial_\nu \{\kappa_0 T_\mu{}^\nu + \vartheta_\mu{}^\nu\} = 0 \tag{3.5}$$

which follows from the field equations (2.4a) just as from the equivalent equations (2.5) and (2.6) is equivalent to

$$\partial_\nu \{\kappa_0(\overset{(\mathrm{tot})}{\mathscr{T}}{}_\mu{}^\nu - \overset{(\mathrm{in})}{\mathscr{T}}{}_\mu{}^\nu) - \tilde{t}_\mu{}^\nu\} = 0 \tag{3.6a}$$

with

$$\tilde{t}_\mu{}^\nu = t_\mu{}^\nu - \overset{(\mathrm{in})}{t}_\mu{}^\nu(g_{\lambda\kappa}) \tag{3.6b}$$

Because the field equations (2.6) and the conservation law (3.5) are equivalent to (2.6) and the gauge (2.5) [if one supposes some asymptotic behavior of the $\psi_{\mu\nu}$, see the theorem of uniqueness of the wave equation as had been proved by Fock (1960)] one can conclude that the field equations (2.6) and the conservation law (3.6) are equivalent to equations (2.5) and (2.6) (and, respectively, to the field equations (2.4)).

4. *Calculation of $g^{\mu\nu}$ and $(-g)^{1/2}$*

In view of the conservation law (3.6) that will be taken later as the equation of motion one has to calculate the contravariant field $g^{\mu\nu}$ and $(-g)^{1/2}$ in terms of the background field $\overset{(\mathrm{in})}{\gamma}_{\mu\nu}$ and of the field

$$h_{\mu\nu} = \sum_{M=1}^{\infty} \underset{(M)}{h_{\mu\nu}} \tag{4.1}$$

From equation (2.1c) one obtains the relation

$$\overset{(\mathrm{in})}{\sigma}{}^{\mu\nu} = -\overset{(\mathrm{in})}{\gamma}{}^{\mu\nu} - \overset{(\mathrm{in})}{\gamma}{}^{\mu\lambda}\overset{(\mathrm{in})}{\sigma}{}_{\lambda}{}^{\nu} \tag{4.2}$$

by which one can calculate successively the contravariant field $\overset{(\mathrm{in})}{\sigma}{}^{\mu\nu}$ in terms of the covariant field $\overset{(\mathrm{in})}{\gamma}_{\mu\nu}$ [because of (1.2b)]. One obtains

$$\overset{(\mathrm{in})}{\sigma}{}^{\mu\nu} = \sum_{N=1}^{\infty} \underset{(N)}{\overset{(\mathrm{in})}{\sigma}{}^{\mu\nu}} \tag{4.3a}$$

with

$$\underset{(1)}{\overset{(\mathrm{in})}{\sigma}{}^{\mu\nu}} = -\overset{(\mathrm{in})}{\gamma}{}^{\mu\nu}$$

$$\underset{(N)}{\overset{(\mathrm{in})}{\sigma}{}^{\mu\nu}} = (-1)^N \overset{(\mathrm{in})}{\gamma}{}^{\mu}{}_{\lambda_1}\overset{(\mathrm{in})}{\gamma}{}^{\lambda_1}{}_{\lambda_2} \cdots \overset{(\mathrm{in})}{\gamma}{}^{\lambda_{N-1}\nu}, \qquad N \geqslant 2 \tag{4.3b}$$

Furthermore it follows from $g_{\mu\nu}g^{\nu\lambda} = \delta_\mu{}^\lambda$ and equations (2.1) that

$$k^{\mu\nu} = -h^{\mu\nu} - h_\lambda{}^\mu \overset{(\mathrm{in})}{\sigma}{}^{\nu\lambda} - \overset{(\mathrm{in})}{\gamma}{}^{\mu\lambda}k_\lambda{}^\nu - h^{\mu\lambda}k_\lambda{}^\nu \tag{4.4}$$

Equation (4.4) shows that $k^{\mu\nu}$ is a mixture of all orders $\underset{(M)}{h_{\mu\nu}}$ and any powers of the background $\overset{(\mathrm{in})}{\gamma}_{\mu\nu}$. So it is useful to define quantities $\underset{(M,N)}{k^{\mu\nu}}$ which are composed of products of the orders $\underset{(M_1)}{h_{\mu\nu}}, \cdots, \underset{(M_n)}{h_{\mu\nu}}$ so that

$$\sum_{\nu=1}^{n} M_\nu = M$$

multiplied by N powers of the background $\overset{(\mathrm{in})}{\gamma}_{\mu\nu}$. Substituting (4.1), (4.3), and

$$k^{\mu\nu} = \sum_{M=1,N=0}^{\infty} \underset{(M,N)}{k^{\mu\nu}} \tag{4.5}$$

into equation (4.4) one obtains

$$\sum_{M=1,N=0}^{\infty} \underset{(M,N)}{k^{\mu\nu}} = -\sum_{M=1}^{\infty} \underset{(M)}{h^{\mu\nu}} - \sum_{M=1,N=1}^{\infty} \underset{(M)}{h^{\mu\lambda}} \underset{(N)}{\overset{(\mathrm{in})}{\sigma}{}_\lambda{}^\nu}$$

$$-\overset{(\mathrm{in})}{\gamma}{}^{\mu\lambda} \sum_{M=1,N=0}^{\infty} \underset{(M,N)}{k_\lambda{}^\nu} - \sum_{M=1}^{\infty} \sum_{L=1,N=0}^{\infty} \underset{(M)}{h^{\mu\lambda}} \underset{(L,N)}{k_\lambda{}^\nu} \tag{4.6}$$

and from this the recurrence relations

$$\underset{(M,0)}{k}^{\mu\nu} = -\underset{(M)}{h}^{\mu\nu} - \sum_{M_1+M_2=M} \underset{(M_1)}{h}^{\mu\lambda} \underset{(M_2,0)}{k_\lambda}^{\nu} \tag{4.7a}$$

$$\underset{(M,1)}{k}^{\mu\nu} = \overset{(\mathrm{in})}{\gamma}{}^{\nu\lambda} \underset{(M)}{h_\lambda}^{\mu} - \overset{(\mathrm{in})}{\gamma}{}^{\mu\lambda} \underset{(M,0)}{k_\lambda}^{\nu} - \sum_{M_1+M_2=M} \underset{(M_1)}{h}^{\mu\lambda} \underset{(M_2,1)}{k_\lambda}^{\nu} \tag{4.7b}$$

$$\underset{(M,N)}{k}^{\mu\nu} = -\underset{(M)}{h}^{\mu\lambda} \underset{(N)}{\overset{(\mathrm{in})}{\sigma_\lambda}}{}^{\nu} - \overset{(\mathrm{in})}{\gamma}{}^{\mu\lambda} \underset{(M,N-1)}{k_\lambda}^{\nu}$$

$$- \sum_{M_1+M_2=M} \underset{(M_1)}{h}^{\mu\lambda} \underset{(M_2,N)}{k_\lambda}^{\nu}, \qquad N \geqslant 2 \tag{4.7c}$$

Starting with

$$\underset{(1,0)}{k}^{\mu\nu} = \underset{(1)}{h}^{\mu\nu} \tag{4.8}$$

one can calculate from these relations all $\underset{(M,N)}{k}^{\mu\nu}$ in terms of the background $\overset{(\mathrm{in})}{\gamma}_{\mu\nu}$ and the $\underset{(M)}{h}_{\mu\nu}$. From the relations (4.7) one obtains the following structure of the $\underset{(M,N)}{k}^{\mu\nu}$:

$$\underset{(M,N)}{k}^{\mu\nu} = {}^{(L)}\underset{(M,N)}{k}^{\mu\nu}(\underset{(M)}{h}_{\kappa\lambda}, \overset{(\mathrm{in})}{\gamma}_{\kappa\lambda}{}^{N}) + {}^{(Q)}\underset{(M,N)}{k}^{\mu\nu}(\underset{(1)}{h}_{\kappa\lambda}, \cdots, \underset{(M-1)}{h}_{\kappa\lambda}, \overset{(\mathrm{in})}{\gamma}_{\kappa\lambda}{}^{N}) \tag{4.9a}$$

$${}^{(L)}\underset{(M,0)}{k} = -\underset{(M)}{h}^{\mu\nu} \tag{4.9b}$$

Here the frontal index (L) means that the quantity depends on the Mth order $\underset{(M)}{h}_{\kappa\lambda}$ linearly, and the index (Q) that the quantity depends on the orders $\underset{(1)}{h}_{\kappa\lambda}, \ldots, \underset{(M-1)}{h}_{\kappa\lambda}$ in form of products of order M; $\overset{(\mathrm{in})}{\gamma}_{\mu\nu}{}^{N}$ stands for N powers of the background $\overset{(\mathrm{in})}{\gamma}_{\mu\nu}$. At fixed N the structure of the ${}^{(L)}\underset{(M,N)}{k}^{\mu\nu}$ is the same for all M. To calculate the determinant g the relation

$$g = \sum_{\alpha,\beta,\gamma,\delta=1}^{4} g_{1\alpha} g_{2\beta} g_{3\gamma} g_{4\delta}\, \epsilon_{\alpha\beta\gamma\delta} \tag{4.10}$$

is used. Here $\epsilon_{\alpha\beta\gamma\delta}$ is the four-dimensional Levi-Civita symbol. From

$$g_{\mu\nu} = \eta_{\mu\nu} + \overset{(\mathrm{in})}{\gamma}_{\mu\nu} + \sum_{M=1}^{\infty} \underset{(M)}{h}_{\mu\nu}$$

then it follows that

$$g = \sum_{M=0, N=0}^{\infty} \underset{(M,N)}{g} \tag{4.11}$$

with

$$\underset{(0,0)}{g} = -1, \qquad \underset{(0,1)}{g} = -\overset{(\mathrm{in})}{\gamma}, \qquad \underset{(0,2)}{g} = \tfrac{1}{2}\left(\overset{(\mathrm{in})}{\gamma}_{\mu\nu}\overset{(\mathrm{in})}{\gamma}^{\mu\nu} - \overset{(\mathrm{in})}{\gamma}{}^{2}\right), \ldots, \underset{(0,4)}{g} = \det \overset{(\mathrm{in})}{\gamma}_{\mu\nu}$$

$$\underset{(0,N)}{g} = 0 \text{ for } N > 4, \quad \sum_{N=0}^{4} \underset{(0,N)}{g} = \det g_{\mu\nu} \tag{4.12a}$$

and

$$\underset{(1,0)}{g} = -\underset{(1)}{h}, \quad \underset{(2,0)}{g} = -\underset{(2)}{h} + \tfrac{1}{2}\left(\underset{(1)}{h}_{\mu\nu}\underset{(1)}{h}^{\mu\nu} - \underset{(1)}{h}{}^{2}\right), \quad \underset{(1,1)}{g} = \underset{(1)}{h}_{\mu\nu}\overset{(\mathrm{in})}{\gamma}^{\mu\nu} - \underset{(1)}{h}\overset{(\mathrm{in})}{\gamma} \tag{4.12b}$$

etc.

The expansion of $(-g)^{1/2}$ according to

$$\begin{aligned}(-g)^{1/2} &= 1 - \tfrac{1}{2}\left(\sum_{N=1}^{4} \underset{(0,N)}{g} + \sum_{M=1,N=0}^{\infty} \underset{(M,N)}{g}\right) \\ &\quad - \tfrac{1}{8}\left(\sum_{N=1}^{4} \underset{(0,N)}{g} + \sum_{M=1,N=0}^{\infty} \underset{(M,N)}{g}\right)^{2} \\ &\quad - \tfrac{1}{16}\left(\sum_{N=1}^{4} \underset{(0,N)}{g} + \sum_{M=1,N=0}^{\infty} \underset{(M,N)}{g}\right)^{3} \\ &\quad \vdots \\ &= \sum_{M=0,N=0}^{\infty} \underset{(M,N)}{(-g)^{1/2}}\end{aligned} \tag{4.13}$$

yields

$$\underset{(0,0)}{(-g)^{1/2}} = 1, \qquad \underset{(0,1)}{(-g)^{1/2}} = \tfrac{1}{2}\overset{(\mathrm{in})}{\gamma}$$

$$\underset{(0,2)}{(-g)^{1/2}} = \tfrac{1}{4}\left(\tfrac{1}{2}\overset{(\mathrm{in})}{\gamma}{}^{2} - \overset{(\mathrm{in})}{\gamma}_{\mu\nu}\overset{(\mathrm{in})}{\gamma}^{\mu\nu}\right) \tag{4.14a}$$

$$\underset{(1,0)}{(-g)^{1/2}} = \tfrac{1}{2}\underset{(1)}{h}, \quad \underset{(2,0)}{(-g)^{1/2}} = \tfrac{1}{2}\underset{(2)}{h} + \tfrac{1}{4}\left(\tfrac{1}{2}\underset{(1)}{h}{}^{2} - \underset{(1)}{h}_{\mu\nu}\underset{(1)}{h}^{\mu\nu}\right)$$

$$\underset{(1,1)}{(-g)^{1/2}} = \tfrac{1}{2}\left(\tfrac{1}{2}\underset{(1)}{h}\overset{(\mathrm{in})}{\gamma} - \underset{(1)}{h}_{\mu\nu}\overset{(\mathrm{in})}{\gamma}^{\mu\nu}\right) \tag{4.14b}$$

etc.

Finally one obtains

$$\mathscr{G}^{\mu\nu} = (-g)^{1/2} g^{\mu\nu} = \sum_{M=0,N=0}^{\infty} \underset{(M,N)}{\mathscr{G}^{\mu\nu}} \tag{4.15}$$

with

$$\underset{(0,0)}{\mathscr{G}^{\mu\nu}} = \eta^{\mu\nu}, \qquad \underset{(0,1)}{\mathscr{G}^{\mu\nu}} = -\overset{(\mathrm{in})}{\gamma}{}^{\mu\nu} + \tfrac{1}{2}\eta^{\mu\nu}\overset{(\mathrm{in})}{\gamma}$$

$$\underset{(0,2)}{\mathscr{G}^{\mu\nu}} = \tfrac{1}{4}\eta^{\mu\nu}(\tfrac{1}{2}\overset{(\mathrm{in})}{\gamma}{}^2 - \overset{(\mathrm{in})}{\gamma}_{\rho\sigma}\overset{(\mathrm{in})}{\gamma}{}^{\rho\sigma}) - \tfrac{1}{2}\overset{(\mathrm{in})}{\gamma}\,\overset{(\mathrm{in})}{\gamma}{}^{\mu\nu} + \overset{(\mathrm{in})}{\gamma}{}^{\mu\lambda}\overset{(\mathrm{in})}{\gamma}_{\lambda}{}^{\nu} \tag{4.16a}$$

$$\underset{(1,0)}{\mathscr{G}^{\mu\nu}} = -\underset{(1)}{h}{}^{\mu\nu} + \tfrac{1}{2}\eta^{\mu\nu}\underset{(1)}{h}$$

$$\underset{(2,0)}{\mathscr{G}^{\mu\nu}} = \tfrac{1}{2}(\underset{(2)}{h} + \tfrac{1}{4}\underset{(1)}{h}{}^2 - \tfrac{1}{2}\underset{(1)}{h}_{\rho\sigma}\underset{(1)}{h}{}^{\rho\sigma})\eta^{\mu\nu} - \tfrac{1}{2}\underset{(1)}{h}\,\underset{(1)}{h}{}^{\mu\nu} \tag{4.16b}$$

$$+ \underset{(1)}{h}{}^{\mu\lambda}\underset{(1)}{h}_{\lambda}{}^{\nu} - \underset{(2)}{h}{}^{\mu\nu}$$

$$\underset{(1,1)}{\mathscr{G}^{\mu\nu}} = \tfrac{1}{2}(\tfrac{1}{2}\underset{(1)}{h}\overset{(\mathrm{in})}{\gamma} - \underset{(1)}{h}_{\rho\sigma}\overset{(\mathrm{in})}{\gamma}{}^{\rho\sigma})\eta^{\mu\nu}$$

$$- \tfrac{1}{2}(\underset{(1)}{h}\overset{(\mathrm{in})}{\gamma}{}^{\mu\nu} + \overset{(\mathrm{in})}{\gamma}\underset{(1)}{h}{}^{\mu\nu}) + \overset{(\mathrm{in})}{\gamma}{}^{\nu\lambda}\underset{(1)}{h}_{\lambda}{}^{\mu} + \overset{(\mathrm{in})}{\gamma}{}^{\mu\lambda}\underset{(1)}{h}{}^{\nu}{}_{\lambda} \tag{4.16c}$$

etc.

5. *General Solution of the Field Equations*

From equations (2.4b), (2.2b), and (4.9) one obtains the following structure of the nonlinear part $\vartheta_{\mu\nu}$:

$$\vartheta_{\mu\nu} = \sum_{M=1,N=0}^{\infty} \underset{(M,N)}{{}^{(Q)}\vartheta_{\mu\nu}}(\underset{(1)}{h}_{\kappa\lambda}, \ldots, \underset{(M-1)}{h}_{\kappa\lambda}, \overset{(\mathrm{in})}{\gamma}_{\kappa\lambda}{}^{N})$$

$$+ \sum_{M=1,N=1}^{\infty} \underset{(M,N)}{{}^{(L)}\vartheta_{\mu\nu}}(\underset{(M)}{h}_{\kappa\lambda}, \overset{(\mathrm{in})}{\gamma}_{\kappa\lambda}{}^{N}) \tag{5.1}$$

For the following calculations it is very important that the structure of

$$\sum_{N=1}^{\infty} \underset{(M,N)}{{}^{(L)}\vartheta_{\mu\nu}}(\underset{(M)}{h}_{\kappa\lambda}, \overset{(\mathrm{in})}{\gamma}_{\kappa\lambda}{}^{N}) \tag{5.2}$$

is the same for all M, i.e.,

$$\sum_{N=1}^{\infty} \underset{(M,N)}{{}^{(L)}\vartheta_{\mu\nu}}(\underset{(M)}{h}_{\kappa\lambda}, \overset{(\mathrm{in})}{\gamma}_{\kappa\lambda}{}^{N}) = \sum_{N=1}^{\infty} \underset{(1,N)}{{}^{(L)}\vartheta_{\mu\nu}}(\underset{(M)}{h}_{\kappa\lambda}, \overset{(\mathrm{in})}{\gamma}_{\kappa\lambda}{}^{N}) \tag{5.3}$$

for all M.

Understanding the series expansion (4.1) as an expansion in powers of κ_0 and using (5.1) one obtains from the field equations (2.6) for the Mth order ($M \geqslant 1$)

$$\Box \underset{(M)}{\psi_{\mu\nu}} = -2\kappa_0 \sum_{N=0}^{\infty} \underset{(M-1,N)}{T_{\mu\nu}} + \sum_{N=0}^{\infty} \underset{(M,N)}{{}^{(Q)}\vartheta_{\mu\nu}}\Big(\underset{(1)}{h_{\kappa\lambda}}, \ldots, \underset{(M-1)}{h_{\kappa\lambda}}, \overset{(\mathrm{in})}{\gamma_{\kappa\lambda}}{}^{N}\Big) + \sum_{N=1}^{\infty} \underset{(M,N)}{{}^{(L)}\vartheta_{\mu\nu}}\Big(\underset{(M)}{h_{\kappa\lambda}}, \overset{(\mathrm{in})}{\gamma_{\kappa\lambda}}{}^{N}\Big) \tag{5.4}$$

Here the matter tensor of order M, i.e.,

$$\sum_{N=0}^{\infty} \underset{(M,N)}{T_{\mu\nu}}$$

depends, on the one hand, on the background field $\overset{(\mathrm{in})}{\gamma_{\mu\nu}}$ and the orders $\underset{(1)}{h_{\mu\nu}}, \ldots, \underset{(M)}{h_{\mu\nu}}$ explicitly. On the other hand, it depends on the background and the orders $\underset{(1)}{h_{\mu\nu}}, \ldots \underset{(M)}{h_{\mu\nu}}$ implicitly because of the equation of motion which has to be satisfied by the matter source $T_{\mu\nu}$ (see below).

If one knows the orders $\underset{(1)}{h_{\mu\nu}}, \ldots, \underset{(M-1)}{h_{\mu\nu}}$ and the matter tensor in the (M-1)th order equation (5.4) appears as a linear inhomogeneous differential equation for the Mth order $\underset{(M)}{h_{\mu\nu}} = \underset{(M)}{\psi_{\mu\nu}} - \frac{1}{2}\eta_{\mu\nu}\underset{(M)}{\psi_{\kappa\lambda}}\eta^{\kappa\lambda}$. Therefore the general solution is the sum of the general solution of the homogeneous equation

$$\Box \underset{(M)}{\psi_{\mu\nu}} = \sum_{N=1}^{\infty} \underset{(M,N)}{{}^{(L)}\vartheta_{\mu\nu}}\Big(\underset{(M)}{h_{\mu\nu}}, \overset{(\mathrm{in})}{\gamma_{\mu\nu}}{}^{N}\Big) \tag{5.5}$$

and a special solution of the inhomogeneous equation. The structure of the homogeneous equation is the same for all M. Its solutions are such without a source $T_{\mu\nu}$ and must to be set zero for physical reasons. A special solution of equation (5.4) is given by

$$\underset{(M)}{\psi_{\mu\nu}} = \overset{(\mathrm{ret})}{\mathscr{I}_1}\left[-2\kappa_0 \sum_{N=0}^{\infty} \underset{(M-1,N)}{T_{\mu\nu}} + \sum_{N=0}^{\infty} \underset{(M,N)}{{}^{(Q)}\vartheta_{\mu\nu}}\Big(\underset{(1)}{h_{\kappa\lambda}}, \ldots, \underset{(M-1)}{h_{\kappa\lambda}}, \overset{(\mathrm{in})}{\gamma_{\kappa\lambda}}{}^{N}\Big)\right] + \sum_{l=2}^{\infty} \overset{(\mathrm{ret})}{\mathscr{I}_l}\left[\sum_{N=1}^{\infty} \underset{(M,N)}{{}^{(L)}\vartheta_{\mu\nu}}\left(\underset{(M)}{h_{\kappa\lambda}}\big(\overset{(\mathrm{ret})}{\mathscr{I}}_{l-1}\big), \overset{(\mathrm{in})}{\gamma_{\kappa\lambda}}{}^{N}\right)\right] \tag{5.6a}$$

Here $\overset{(\mathrm{ret})}{\mathscr{I}_1}[\cdots]$ means the retarded integral $\overset{(\mathrm{ret})}{\mathscr{I}} = \overset{(\mathrm{ret})}{\Box^{-1}}$ of the quantity in the square brackets and

$$\underset{(M)}{h_{\mu\nu}}\big(\overset{(\mathrm{ret})}{\mathscr{I}}_{l-1}\big) = \underset{(M)}{\psi_{\mu\nu}}\big(\overset{(\mathrm{ret})}{\mathscr{I}}_{l-1}\big) - \tfrac{1}{2}\eta_{\mu\nu}\underset{(M)}{\psi_{\kappa\lambda}}\big(\overset{(\mathrm{ret})}{\mathscr{I}}_{l-1}\big)\eta^{\kappa\lambda} \tag{5.6b}$$

The solution (5.6) supposes the existence of the several retarded integrals and the convergence of the series. Then, in fact, it follows that

$$\begin{aligned}
\underset{(M)}{\Box\psi_{\mu\nu}} &= -2\kappa_0 \sum_{N=0}^{\infty} \underset{(M-1,N)}{T_{\mu\nu}} + \sum_{N=0}^{\infty} \underset{(M,N)}{{}^{(Q)}\vartheta_{\mu\nu}}(\cdots) \\
&\quad + \sum_{l=1}^{\infty}\sum_{N=1}^{\infty} \underset{(M,N)}{{}^{(L)}\vartheta_{\mu\nu}}\left(\underset{(M)}{h_{\kappa\lambda}}(\overset{(\mathrm{ret})}{\mathscr{I}_l})\overset{(\mathrm{in})}{\gamma_{\kappa\lambda}}{}^{N}\right) \\
&= -2\kappa_0 \sum_{N=0}^{\infty} \underset{(M-1,N)}{T_{\mu\nu}} + \sum_{N=0}^{\infty} \underset{(M,N)}{{}^{(Q)}\vartheta_{\mu\nu}}(\cdots) \\
&\quad + \sum_{N=1}^{\infty} \underset{(M,N)}{{}^{(L)}\vartheta_{\mu\nu}}\Bigg(\underbrace{\sum_{l=1}^{\infty} \underset{(M)}{h_{\kappa\lambda}}(\overset{(\mathrm{ret})}{\mathscr{I}_l})}_{\underset{(M)}{h_{\kappa\lambda}}}, \overset{(\mathrm{in})}{\gamma_{\kappa\lambda}}{}^{N}\Bigg)
\end{aligned}$$

The solution (5.6) shows the structure

$$\underset{(M)}{\psi_{\mu\nu}} = \sum_{N=0}^{\infty} \underset{(M)}{\overset{(N)}{\psi_{\mu\nu}}} \tag{5.7}$$

of the $\underset{(M)}{\psi_{\mu\nu}}$. That means that $\underset{(M)}{\overset{(N)}{\psi_{\mu\nu}}}$ is of order $\overset{(\mathrm{in})}{\gamma}{}^{N}\underset{(M)}{\overset{(0)}{\psi_{\mu\nu}}}$. It holds

$$\underset{(M)}{\overset{(0)}{\psi_{\mu\nu}}} = \overset{(\mathrm{ret})}{\mathscr{I}_1}\,[-2\kappa_0 \underset{(M-1,0)}{T_{\mu\nu}} + \underset{(M,0)}{{}^{(Q)}\vartheta_{\mu\nu}}(\underset{(1)}{h_{\kappa\lambda}}, \ldots, \underset{(M-1)}{h_{\kappa\lambda}})] \tag{5.8a}$$

$$\begin{aligned}
\underset{(M)}{\overset{(1)}{\psi_{\mu\nu}}} &= \overset{(\mathrm{ret})}{\mathscr{I}_1}\,[-2\kappa_0 \underset{(M-1,1)}{T_{\mu\nu}} + \underset{(M,1)}{{}^{(Q)}\vartheta_{\mu\nu}}(\underset{(1)}{h_{\kappa\lambda}}, \ldots, \underset{(M-1)}{h_{\kappa\lambda}}, \overset{(\mathrm{in})}{\gamma_{\kappa\lambda}})] \\
&\quad + \overset{(\mathrm{ret})}{\mathscr{I}_2}\,[\underset{(M,1)}{{}^{(L)}\vartheta_{\mu\nu}}(\underset{(M)}{\overset{(0)}{h_{\kappa\lambda}}}, \overset{(\mathrm{in})}{\gamma_{\kappa\lambda}})]
\end{aligned} \tag{5.8b}$$

$$\begin{aligned}
\underset{(M)}{\overset{(2)}{\psi_{\mu\nu}}} &= \overset{(\mathrm{ret})}{\mathscr{I}_1}\,[-2\kappa_0 \underset{(M-1,2)}{T_{\mu\nu}} + \underset{(M,2)}{{}^{(Q)}\vartheta_{\mu\nu}}(\underset{(1)}{h_{\kappa\lambda}}, \ldots, \underset{(M-1)}{h_{\kappa\lambda}}, \overset{(\mathrm{in})}{\gamma}{}^{2}_{\kappa\lambda})] \\
&\quad + \overset{(\mathrm{ret})}{\mathscr{I}_2}\,[\underset{(M,2)}{{}^{(L)}\vartheta_{\mu\nu}}(\underset{(M)}{\overset{(0)}{h_{\kappa\lambda}}}, \overset{(\mathrm{in})}{\gamma}{}^{2}_{\kappa\lambda}) + \underset{(M,1)}{{}^{(L)}\vartheta_{\mu\nu}}(\underset{(M)}{\overset{(1)}{h_{\kappa\lambda}}}(\overset{(\mathrm{ret})}{\mathscr{I}_1}), \overset{(\mathrm{in})}{\gamma_{\kappa\lambda}})] \\
&\quad + \overset{(\mathrm{ret})}{\mathscr{I}_3}\,[\underset{(M,1)}{{}^{(L)}\vartheta}\,(\underset{(M)}{\overset{(1)}{h_{\kappa\lambda}}}(\overset{(\mathrm{ret})}{\mathscr{I}_2}), \overset{(\mathrm{in})}{\gamma_{\kappa\lambda}})]
\end{aligned} \tag{5.8c}$$

and in general

$$
\begin{aligned}
\underset{(M)}{\overset{(N)}{\psi_{\mu\nu}}} = {} & \overset{(\text{ret})}{\mathscr{I}_1}\Big[-2\kappa_0 \underset{(M-1,N)}{T_{\mu\nu}} + \underset{(M,N)}{{}^{(Q)}\vartheta_{\mu\nu}}\big(\underset{(1)}{h_{\kappa\lambda}}, \ldots, \underset{(M-1)}{h_{\kappa\lambda}}, \overset{(\text{in})}{\gamma_{\kappa\lambda}}{}^{N}\big)\Big] \\
& + \overset{(\text{ret})}{\mathscr{I}_2}\Big[\underset{(M,N)}{{}^{(L)}\vartheta_{\mu\nu}}\big(\underset{(M)}{\overset{(0)}{h_{\kappa\lambda}}}, \overset{(\text{in})}{\gamma_{\kappa\lambda}}{}^{N}\big) + \underset{(M,N-1)}{{}^{(L)}\vartheta_{\mu\nu}}\big(\underset{(M)}{\overset{(1)}{h_{\kappa\lambda}}}(\overset{(\text{ret})}{\mathscr{I}_1}), \overset{(\text{in})}{\gamma_{\kappa\lambda}}{}^{N}\big) \\
& + \cdots \underset{(M,1)}{{}^{(L)}\vartheta_{\mu\nu}}\big(\underset{(M)}{\overset{(N-1)}{h_{\kappa\lambda}}}(\overset{(\text{ret})}{\mathscr{I}_1}), \overset{(\text{in})}{\gamma_{\kappa\lambda}}\big)\Big] \qquad (5.8\text{d}) \\
& + \overset{(\text{ret})}{\mathscr{I}_3}\Big[\underset{(M,N-1)}{{}^{(L)}\vartheta_{\mu\nu}}\big(\underset{(M)}{\overset{(1)}{h_{\kappa\lambda}}}(\overset{(\text{ret})}{\mathscr{I}_2}), \overset{(\text{in})}{\gamma_{\kappa\lambda}}{}^{N-1}\big) + \cdots + \underset{(M,1)}{{}^{(L)}\vartheta_{\mu\nu}}\big(\underset{(M)}{\overset{(N-1)}{h_{\kappa\lambda}}}(\overset{(\text{ret})}{\mathscr{I}_2}), \overset{(\text{in})}{\gamma_{\kappa\lambda}}\big)\Big] \\
& \qquad \vdots \\
& + \overset{(\text{ret})}{\mathscr{I}_{N+1}}\Big[\underset{(M,1)}{{}^{(L)}\vartheta_{\mu\nu}}\big(\underset{(M)}{\overset{(N-1)}{h_{\kappa\lambda}}}(\overset{(\text{ret})}{\mathscr{I}_N}), \overset{(\text{in})}{\gamma_{\kappa\lambda}}\big)\Big]
\end{aligned}
$$

6. *Successive Procedure to Solve the Field Equations and the Equation of Motion*

The equation of motion is used in form of the conservation law (3.6). In space-time regions with $\overset{(\text{in})}{T_{\mu\nu}} = 0$ this conservation law takes the form

$$\partial_\nu \{\kappa_0 \mathscr{G}^{\nu\lambda} T_{\mu\lambda} - \tilde{t}_\mu{}^\nu\} = 0 \tag{6.1}$$

Because $\tilde{t}_\mu{}^\nu$ is nonlinear in the field just as $\vartheta_{\mu\nu}$ one has

$$\tilde{t}_\mu{}^\nu = \sum_{M=1,N=0}^{\infty} \underset{(M,N)}{{}^{(Q)}\tilde{t}_\mu{}^\nu}\big(\underset{(1)}{h_{\kappa\lambda}}, \ldots, \underset{(M-1)}{h_{\kappa\lambda}}, \overset{(\text{in})}{\gamma_{\kappa\lambda}}{}^{N}\big) + \sum_{M=1,N=1}^{\infty} \underset{(M,N)}{{}^{(L)}\tilde{t}_\mu{}^\nu}\big(\underset{(M)}{h_{\kappa\lambda}}, \overset{(\text{in})}{\gamma_{\kappa\lambda}}{}^{N}\big) \tag{6.2}$$

Using the series expansion (4.15) and (4.16) of $\mathscr{G}^{\mu\nu}$ the equation of motion (6.1) in the Mth order then takes the form

$$
\begin{aligned}
\partial_\nu \Big\{\kappa_0 \sum_{N=0}^{\infty} \Big(& \underset{(M-1,N)}{T_\mu{}^\nu} + \sum_{\substack{N_1+N_2=N \\ N_2 \leqslant N-1}} \underset{(0,N_1)}{\mathscr{G}^{\nu\lambda}} \underset{(M-1,N_2)}{T_{\mu\lambda}} \\
& + \sum_{\substack{M_1+M_2=M-1 \\ M_2 \leqslant M-2}} \sum_{N_1+N_2=N} \underset{(M_1,N_1)}{\mathscr{G}^{\nu\lambda}} \underset{(M_2,N_2)}{T_{\mu\lambda}}\Big) \qquad (6.3) \\
& - \sum_{N=0}^{\infty} \underset{(M,N)}{{}^{(Q)}\tilde{t}_\mu{}^\nu}(\cdots) - \sum_{N=1}^{\infty} \underset{(M,N)}{{}^{(L)}\tilde{t}_\mu{}^\nu}(\cdots)\Big\} = 0, \qquad M \geqslant 1
\end{aligned}
$$

Knowing the orders $\underset{(1)}{h}_{\kappa\lambda}, \ldots, \underset{(M-1)}{h}_{\kappa\lambda}$ and the matter tensor up to the $(M$-$2)$th order one can calculate the Mth order $\underset{(M)}{h}_{\kappa\lambda}$ and the $(M$-$1)$th order of the matter tensor from the solution (5.8) of the field equations and the equation of motion (6.3).

To do this, start from equation (6.3) for $N = 0$

$$\partial_\nu \Big\{\kappa_0 \Big(\underset{(M-1,\,0)}{T_\mu{}^\nu} + \sum_{\substack{M_1+M_2=M-1\\ M_2 \leqslant M-2}} \underset{(M_1,\,0)}{\mathcal{G}^{\nu\lambda}} \underset{(M_2,\,0)}{T_{\mu\lambda}}\Big) - \underset{(M,\,0)}{{}^{(Q)}\tilde{t}_\mu{}^\nu}\big(\underset{(1)}{h}_{\kappa\lambda}, \ldots, \underset{(M-1)}{h}_{\kappa\lambda}\big)\Big\} = 0 \tag{6.4}$$

By solving equation (6.4) one obtains $\underset{(M-1,\,0)}{T_{\mu\nu}}$ and by that the field $\overset{(0)}{\underset{(M)}{h}}_{\mu\nu}$ from equation (5.8a). For $N = 1$ equation (6.3) takes the form

$$\begin{aligned}\partial_\nu \Big\{\kappa_0 \Big(&\underset{(M-1,\,1)}{T_\mu{}^\nu} + \underset{(0,\,1)}{\mathcal{G}^{\nu\lambda}} \underset{(M-1,\,0)}{T_{\mu\lambda}} + \sum_{\substack{M_1+M_2=M-1\\ M_2 \leqslant M-2}} \sum_{N_1+N_2=1} \underset{(M_1,\,N_1)}{\mathcal{G}^{\nu\lambda}} \underset{(M_2,\,N_2)}{T_{\mu\lambda}}\Big)\\ &- \underset{(M,\,1)}{{}^{(Q)}\tilde{t}_\mu{}^\nu}\big(\underset{(1)}{h}_{\kappa\lambda}, \ldots, \underset{(M-1)}{h}_{\kappa\lambda}, \overset{(\mathrm{in})}{\gamma}_{\kappa\lambda}\big) - \underset{(M,\,1)}{{}^{(L)}\tilde{t}_\mu{}^\nu}\big(\overset{(0)}{\underset{(M)}{h}}_{\kappa\lambda}, \overset{(\mathrm{in})}{\gamma}_{\kappa\lambda}\big)\Big\} = 0\end{aligned} \tag{6.5}$$

By solving equation (6.5) one obtains $\underset{(M-1,1)}{T_{\mu\nu}}$ and by that the field $\overset{(1)}{\underset{(M)}{h}}_{\mu\nu}$ from equation (5.8b). Using $\overset{(0)}{\underset{(M)}{h}}_{\mu\nu}$ and $\overset{(1)}{\underset{(M)}{h}}_{\mu\nu}$ one gets $\underset{(M-1,\,2)}{T_{\mu\nu}}$ from equation (6.3) for $N = 2$, etc. The essential point of this successive procedure is the smallness of the curved background field so that it is possible to expand each order $\underset{(M)}{h}_{\mu\nu}$ in powers of the background.

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